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Theta Modular Groups Determined by Point Sets.

By Arthur B. Coble.[*]

Introduction.

In a series of articles already published[†] by the writer, the study of the properties of a set P_n^k of n discrete points in S_k has been initiated. In these articles particular attention has been paid to certain discontinuous groups defined by the P_n^k, to the invariants of these groups, and to applications to the theory of equations. During the course of the work there appeared, in a special case, a connection between the point set and theta modular functions. It is the purpose of this article to show that this connection must exist in the general case.

An especially interesting type of point set is the self-associated P_{2p+2}^p.[‡] This set of $2p+2$ points in S_p has the characteristic geometric property that all the quadrics on $2p+1$ points of the set pass through the remaining point. It has the characteristic algebraic property that complementary determinants formed from the matrix of the coordinates of the points differ by a fixed factor of proportionality. If $p+2$ of the points be chosen as a base in S_p, the remaining p points lie on an S_{p-1}, α, in S_p, and quadrics on the base cut α in quadric sections, all of which are apolar to a quadric Q_α in α. The p points on α are any self-polar p-edron of Q_α. After choosing the base in S_p, p absolute constants are required to fix α, and $\frac{1}{2}p(p-1)$ absolute constants are required to fix the self-polar p-edron of Q_α, whence the number of absolute constants in the self-associated P_{2p+2}^p is $\frac{1}{2}p(p+1)$.

Thus the number of absolute constants of this self-associated point set and the number of moduli of the general theta function of p variables are the same. This might be dismissed as a mere coincidence were it not for the fact that the point set and the moduli are known to be related in a number of special cases. A general class of such cases is the set of $2p+2$ points on the rational norm curve R^p in S_p. Obviously all the quadrics on $2p+1$ of these points will pass through the remaining one, and the set is self-associated. Let us call it a *hyperelliptic* self-associated set. It is well known that such a set defines the hyperelliptic theta functions with $2p-1$ independent moduli. Nor is the relation here existing peculiar to the hyperelliptic case. For when p is 3, the smallest value of p for which the theta functions are not necessarily hyperelliptic, it is well known that the self-associated set of eight points in S_3 will define a quartic curve of genus 3, and thereby also the moduli of the theta functions of genus 3. In all of these cases the properties of the point set and of the modular functions have been developed sufficiently to make the connection quite clear.*

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[*] This investigation has been carried on under the auspices of the Carnegie Institution of Washington, D. C.

[†] "Point Sets and Cremona Groups," Parts I, II, III, *Transactions of the American Mathematical Society*, 1915–16–17. These are referred to hereafter as P. S. I, II, III respectively.

[‡] P. S. I (12).

In P. S. I § 6, projectively equivalent ordered sets P_n^k were mapped upon a point P of a space $\Sigma_{k(n-k-2)}$ and by permutation of the points of P_n^k a group $G_{n!}$ in Σ was derived. A conjugate set of points P in Σ under $G_{n!}$ represents all sets P_n^k projective in some order to each other. In P. S. II § 3, the congruence of sets P_n^k was defined. Two point sets P_n^k and $P_n'^k$ are congruent if the one arises from the other by means of a sequence of operations which are either projectivities or those particular Cremona transformations which occur when the variables are inverted. In applying an operation of the latter type it is understood that the $k+1$ F-points of the Cremona transformation belong to both congruent sets P_n^k and $P_n'^k$, and that the remaining $n-(k+1)$ points of each set form corresponding pairs of the Cremona transformation. It then appeared that all point sets congruent in some order to a given set P_n^k were mapped in Σ by points P which formed a conjugate set under a group $G_{n,k}$ in $\Sigma_{k(n-k-2)}$. This infinite, discontinuous Cremona group $G_{n,k}$ is thereby projectively defined by the point set P_n^k. In the main this article is concerned with the structure of this group.

Though generating elements of $G_{n,k}$ have been given (P. S. II(15)), it is more convenient to handle certain isomorphic groups of linear transformations, particularly the group $g_{n,k}$. The operation which transforms a point set P_n^k into a congruent set $P_n'^k$ will transform an algebraic spread in S_k of order x_0, and multiplicities $x_1, \ldots, x_n$ at the points of P_n^k into an algebraic spread of order x_0' with multiplicities $x_1', \ldots, x_n'$ at the points of $P_n'^k$. Then x' is the linear transform of x under an element of $g_{n,k}$. The group $g_{n,k}$ is simply isomorphic with $G_{n,k}$ except in a few particular cases with which we are not concerned here. The elements of $g_{n,k}$ have determinants ± 1 and integer coefficients. Hence the group $g_{n,k}$ admits of a modular theory precisely similar to that of the groups of transformations of the periods of the multiply periodic functions. For example, it is clear that

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* For the case $p=3$, however, explicit formulae for the invariants of the point set in terms of the modular functions are still lacking.

(1) *The elements of $g_{n,k}$ whose coefficients reduce modulo m to those of the identity, form an infinite invariant subgroup of $g_{n,k}$ whose factor group $g^{(m)}_{n,k}$ is finite.*

We shall determine in this article the order and the structure of all the groups $g^{(2)}_{n,k}$. Since $g_{n,k}$ has a quadratic invariant we should expect these groups to be subgroups of the group of a null system in the finite geometry mod. 2.

In § 1 the generators and invariants of $g_{n,k}$ are given, and these are reduced mod. 2, to generators and invariants respectively of $g^{(2)}_{n,k}$. Certain new invariants are derived and the sets of linear forms conjugate under $g^{(2)}_{n,k}$ are tabulated. These linear forms are permuted under $g^{(2)}_{n,k}$ just as certain geometric objects in the finite geometry of the null system are permuted under certain subgroups of the group of the null system. The identification of the forms and the corresponding geometric objects is made in § 3, there being sixteen cases according as

$$(2) \qquad k=4\lambda+\varkappa, \quad n=4\mu+\nu \quad (\varkappa, \nu=0, 1, 2, 3).$$

The finite geometry is exhibited in terms of the basis notation. This notation is recapitulated in § 2, and is there used to determine the structure of the groups which appear in § 3.*

In order to impose conveniently the conditions for a self-associated set, we use the *ultra-elliptic* point set P^k_n, *i. e.*, n-points on the normal elliptic curve E^{k+1} in S_k with elliptic parameters $u_1, \ldots, u_n$. Thus in S_p the $2p+2$-points cut out on E^{p+1} by a quadric evidently constitute an ultra-elliptic self-associated set, and the condition for self-association is merely

$$(3) \qquad u_1+u_2+\ldots+u_{2p+2}\equiv 0.$$

For $p=2$ and $p=3$ every self-associated set is ultra-elliptic, and it can be shown that this is true for $p=4$ also. For further values of p the ultra-elliptic self-associated set must be special since it contains only $2p+2$ absolute constants—one for the E^{p+1} and $2p+1$ for the parameters of all but one of the points.

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* The statements are drawn from two earlier papers of the writer: "An Application of Finite Geometry to the Characteristic Theory of the Odd and Even Theta Functions," *Trans. Amer. Math. Soc.*, Vol. XIV (1913), and "An Isomorphism between Theta Characteristics and the $(2p+2)$-point," *Annals of Math.*, Vol. XVII, Ser. 2 (1916). These are referred to respectively as F. G. I and F. G. II.

If an ultra-elliptic set P_n^k on E^{k+1} be transformed into a congruent ultra-elliptic set $P_n'^k$ on E'^{k+1}, the two elliptic curves are projective. If E'_{k+1} be projected upon E^{k+1} the set $P_n'^k$ is projected upon a set on E^{k+1} with parameters $u'_1, \dots, u'_n$. Then the u's and us are linearly related under an element of the group $e_{n,k}$ which is simply isomorphic with $g_{n,k}$ (P. S. II (33)). Thus, though the ultra-elliptic set is special, it can be used to determine the structure of the groups determined by the general set.

In §4 the relation of $g_{n,k}^{(2)}$ to modular groups determined by $e_{n,k}$ is discussed, and the restriction to self-associated sets P_{2p+2}^p is made by the use of (3).

We shall assume throughout that $n \geqq k+3$.

§1. *The Generators, Invariants and Conjugate Linear Forms of* $g_{n,k}^{(2)}$.

The group $g_{n,k}$ is generated (P. S. II §5) by transpositions such as T_{12} of the variables $x_1, \dots, x_n$ and the involutory element

$$A_{1,2,\dots,k+1}: \begin{array}{ll} x'_i = x_i + [(k-1)x_0 - x_1 - x_2 \dots - x_{k+1}] & (i=0, 1, 2, \dots, k+1), \\ x'_j = x_j \quad (j=k+1, \dots, n). & \end{array}$$

These generators belong in a conjugate set. The group has an invariant quadric form M, and an invariant point O and linear form L which are pole and polar as to M. These are

$$\begin{aligned} M &= (k-1)x_0^2 - (x_1^2 + x_2^2 + \dots + x_n^2), \\ L &= (k+1)x_0 - (x_1 + x_2 + \dots + x_n), \\ O &= k+1,\ k-1,\ k-1, \dots, k-1. \end{aligned}$$

If these generators be reduced mod. 2, and thereafter combined mod. 2, they become the generators of $g_{n,k}^{(2)}$. The above invariants similarly reduced become invariants of $g_{n,k}^{(2)}$.

Thus $g_{n,k}^{(2)}$ is generated by transpositions such as T_{12} and $A_{1,\dots,k+1}$ where

$$(4) \quad \begin{array}{lll} & k \text{ even.} & k \text{ odd.} \\ A_{1,\dots,k+1}: & \begin{array}{l} x'_i = x_i + (x_0 + x_1 + \dots + x_{k+1}), \\ x'_j = x_j, \end{array} & : \begin{array}{l} x'_i = x_i + (x_1 + \dots + x_{k+1}), \\ x'_j = x_j, \end{array} \\ & \multicolumn{2}{l}{(i=0, 1, \dots, k+1;\ j=k+2, \dots, n).} \end{array}$$

Writing M above in polarized form before reducing we have as invariants of $g_{n,k}^{(2)}$:

$$(5) \quad \left\{ \begin{array}{ll} \quad k \text{ even.} & \quad k \text{ odd.} \\ M(x, y) = x_0y_0 + x_1y_1 + \dots + x_ny_n, & M(x, y) = x_1y_1 + \dots + x_ny_n, \\ L = x_0 + x_1 + \dots + x_n, & L = x_1 + \dots + x_n, \\ O = 1, 1, \dots, 1; & O = 0, 1, \dots, 1. \end{array} \right.$$

It may happen that $g^{(2)}_{n,k}$ will have other quadratic invariants. Any quadratic form symmetric in $x_1, \dots, x_n$ after possible subtraction of $M(x, x)$ takes the form

$$\alpha x_0^2+\beta x_0(x_1+\dots+x_n)+\gamma(x_1x_2+\dots+x_{n-1}x_n). \tag{6}$$

For k even let $x_i'=x_i+D$, $x_j'=x_j$ be $A_{1,\dots,k+1}$. Under this element (6) acquires the increment

$$\alpha D^2+\beta\{Dx_0+D(x_1+\dots+x_n)+D^2\}+\gamma\left\{\binom{k+1}{2}D^2+(x_{k+2}+\dots+x_n)D\right\},$$

and this increment must vanish. After factoring out D, we find from the typical terms in x_0, x_1, x_n that $\alpha+\gamma\frac{k}{2}\equiv\beta+\gamma\equiv 0 \bmod. 2$. Hence defining $\varkappa$ as in (2) we have when $\varkappa=0$ that $\alpha=0$, $\beta=\gamma=1$; and when $\varkappa=2$ that $\alpha=\beta=\gamma=1$. When k is odd the increment is

$$\alpha D^2+\beta D(x_1+\dots+x_n)+\gamma\left\{\binom{k+1}{2}D^2+D^2\right\},$$

whence $\alpha+\beta+\left\{\frac{k+1}{2}+1\right\}\gamma\equiv\beta\equiv 0$. If $\varkappa=1$, then $\alpha=\beta=0, \gamma=1$; and if $\varkappa=3$, then $\beta=0$, $\alpha=\gamma=1$. Hence

(7) *The* $g^{(2)}_{n,k}$ *has the additional invariant quadratic form:*

$$\begin{aligned}
\varkappa&=0, & M'&= & &x_0(x_1+\dots+x_n)+(x_1x_2+\dots+x_{n-1}x_n)\,;\\
\varkappa&=2, & M'&=x_0^2 &+&x_0(x_1+\dots+x_n)+(x_1x_2+\dots+x_{n-1}x_n)\,;\\
\varkappa&=1, & M'&= & &(x_1x_2+\dots+x_{n-1}x_n)\,;\\
\varkappa&=3, & M'&=x_0^2 & &+(x_1x_2+\dots+x_{n-1}x_n).
\end{aligned}$$

Thus in all cases the $g^{(2)}_{n,k}$ is simply or multiply isomorphic with the group of a quadric or with a subgroup of such a group. In order to determine it more precisely we shall determine its effect upon the system of integer linear forms whose coefficients are reduced mod. 2, *i. e.*, upon the spaces S_{n-1} in the S_n of $x_0, x_1, \dots, x_n$. Any element of $g^{(2)}_{n,k}$ is completely defined if its effect upon any $n+1$ linearly independent forms is known.

All of these forms are comprised under the following types:

$$B_{1,2,\dots,l}=x_1+x_2+\dots+x_l, \quad C_{1,2\dots,l}=x_0+x_1+x_2+\dots+x_l. \tag{8}$$

We shall denote by $B(l)$ and $C(l)$ respectively the aggregate of forms (8) obtained by taking all sets of l variables from $x_1, \dots, x_n$; and we shall denote further by b_1, b_2, b_3, b_4 and c_0, c_1, c_2, c_3 the aggregate of forms $B(l)$,

$C(l)$ for which $l \equiv 1, 2, 3, 4$ or $l \equiv 0, 1, 2, 3$ mod. 4 respectively. From these aggregates the invariant form

$$\begin{aligned} C_{1,2,\ldots\ldots,n} &= x_0+x_1+\ldots\ldots+x_n \quad (k \text{ even}), \\ B_{1,2,\ldots\ldots,n} &= \quad\; x_1+\ldots\ldots+x_n \quad (k \text{ odd}) \end{aligned}$$

is to be excluded.

Let us consider first the

CASE I: k even.

It is then quite clear that

(9) *A form* $\frac{B(l)}{C(l)}$ *is unaltered, or is transformed into a form* $\frac{C(l+k+1-2r)}{B(l+k+1-2r)}$, *by* $A_{1,\ldots\ldots,k+1}$ *according as* $A_{1,\ldots\ldots,k+1}$ *has an* $\frac{\textit{even}}{\textit{odd}}$ *or* $\frac{\textit{odd}}{\textit{even}}$ *number* r *of subscripts in common with* $\frac{B(l)}{C(l)}$. *If the form is altered, the subscripts of the transformed form are those which belong either to* $A_{1,\ldots\ldots,k+1}$, *or to the original form, but not to both.*

When a B is transformed into a C, and this C back into a B', or a C into a B and this B back into a C', the number of subscripts is increased by $2k+2-2(r+r')$ where r, r' is odd, even or even, odd. Thus forms $B(l), B(l')$ or forms $C(l)$, $C(l')$ are conjugate only when $l \equiv l'$ mod. 4. Moreover, forms $B(l)$, $C(l')$ are conjugate only when $l' \equiv l+k+1-2r$ mod. 4 (r odd). The latter condition depends upon the value of k and we find that

(10) *Under* $g^{(2)}_{n,k}$, *k even, there are four sets of conjugate linear forms, namely:*

$$\begin{aligned} \varkappa=0&: \; b_1, c_0; \; b_2, c_1; \; b_3, c_2; \; b_4, c_3; \\ \varkappa=2&: \; b_1, c_2; \; b_2, c_3; \; b_3, c_0; \; b_4, c_1; \end{aligned}$$

the linear form L being excluded.

That all of the linear forms indicated actually occur in a conjugate set is readily verified. Indeed, beginning with $B(y)$ ($y=1, 2, 3, 4$) we obtain $C(y+k-1)$ and also, if $y=3$ or 4, $C(y+k-5)$. From these we get in turn all the admissible forms $B(y)$, $B(y+4)$, ..., $B(y+2k)$; etc.

Every linear form is paired by addition to the invariant form L with another linear form. These pairs are permuted by $g^{(2)}_{n,k}$ as entities. For k even the pairs of forms are $B(l), C(n-l)$. Recalling from (2) that $n=4\mu+\nu$ ($\nu=0, 1, 2, 3$) we have the following table of conjugate forms and their pairs:

$$(11)\qquad \begin{array}{l}
\qquad\qquad\quad \varkappa=0 \qquad\qquad\qquad\qquad\qquad \varkappa=2 \\
\left\{\begin{array}{l}
\nu=0\left\{\begin{matrix} \begin{matrix} b_1, c_0 \\ c_3, b_4 \end{matrix} & \begin{matrix} b_2, c_1 \\ c_2, b_3 \end{matrix} & \begin{matrix} b_1, c_2 \\ c_3, b_2 \end{matrix} & \begin{matrix} b_3, c_0 \\ c_1, b_4 \end{matrix} \end{matrix}\right. \\
\nu=2\left\{\begin{matrix} \begin{matrix} b_1, c_0 \\ c_1, b_2 \end{matrix} & \begin{matrix} b_3, c_2 \\ c_3, b_4 \end{matrix} & \begin{matrix} b_1, c_2 \\ c_1, b_4 \end{matrix} & \begin{matrix} b_2, c_3 \\ c_0, b_3 \end{matrix} \end{matrix}\right. \\
\nu=1\left\{\begin{matrix} \begin{matrix} b_2, c_1 \\ c_3, b_4 \end{matrix} & \left\{\begin{matrix} b_1 \\ c_0 \end{matrix}\right.\left\{\begin{matrix} b_3 \\ c_2 \end{matrix}\right. & \begin{matrix} b_1, c_2 \\ c_0, b_3 \end{matrix} & \left\{\begin{matrix} b_2 \\ c_3 \end{matrix}\right.\left\{\begin{matrix} b_4 \\ c_1 \end{matrix}\right. \end{matrix}\right. \\
\nu=3\left\{\begin{matrix} \begin{matrix} b_1, c_0 \\ c_2, b_3 \end{matrix} & \left\{\begin{matrix} b_2 \\ c_1 \end{matrix}\right.\left\{\begin{matrix} b_4 \\ c_3 \end{matrix}\right. & \begin{matrix} b_2, c_3 \\ c_1, b_4 \end{matrix} & \left\{\begin{matrix} b_1 \\ c_2 \end{matrix}\right.\left\{\begin{matrix} b_3 \\ c_0 \end{matrix}\right. \end{matrix}\right.
\end{array}\right.
\end{array}$$

Within any compartment conjugate forms are in line, paired forms in column and forms both conjugate and paired are indicated by $\}$.

Turning next to the

CASE II: k odd,

we find that

(12) *A form* $\genfrac{}{}{0pt}{}{B(l)}{C(l)}$ *is unaltered, or is transformed into a form* $\genfrac{}{}{0pt}{}{B(l+k+1-2r)}{C(l+k+1-2r)}$, *by* $A_{1,\ldots,k+1}$ *according as* $A_{1,\ldots,k+1}$ *has an* $\genfrac{}{}{0pt}{}{\textit{even}}{\textit{odd}}$ *or* $\genfrac{}{}{0pt}{}{\textit{odd}}{\textit{even}}$ *number r of subscripts in common with* $\genfrac{}{}{0pt}{}{B(l)}{C(l)}$.

The subscripts of the transformed form are found by the same rule as above. We find in all six sets of conjugate forms and the table analogous to (10) is:

$$(13)\qquad \left\{\begin{array}{l}
\varkappa=1:\ b_1;\ b_2;\ b_3;\ b_4;\ c_0,\ c_2;\ c_1,\ c_3; \\
\varkappa=3:\ b_1,\ b_3;\ b_2,\ b_4;\ c_0;\ c_1;\ c_2;\ c_3.
\end{array}\right.$$

where, as before, the form L is excluded.

On adding the form $L=B_{1\,\ldots\ldots,n}$ the forms $B(l)$, $B(n-l)$ are paired, and also the forms $C(l)$, $C(n-l)$. Hence the table formed, as in (11), of conjugate and paired forms is as follows:

$$(14)\qquad \begin{array}{l}
\qquad\qquad\qquad\qquad \varkappa=1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \varkappa=3 \\
\left\{\begin{array}{l}
\nu=0\left\{\begin{matrix} \begin{matrix} b_1 \\ b_3 \end{matrix} & \begin{matrix} \}b_2 \\ \end{matrix} & \begin{matrix} \}b_4 \\ \end{matrix} & \begin{matrix} \}c_0 \\ \end{matrix} & \begin{matrix} \}c_2 \\ \end{matrix} & \left\{\begin{matrix} c_1 \\ c_3 \end{matrix}\right. & \quad & \left\{\begin{matrix} b_1 \\ b_3 \end{matrix}\right. & \begin{matrix} \}b_2 \\ \end{matrix} & \begin{matrix} \}b_4 \\ \end{matrix} & \begin{matrix} \}c_0 \\ \end{matrix} & \begin{matrix} \}c_2 \\ \end{matrix} & \begin{matrix} c_1 \\ c_3 \end{matrix} \end{matrix}\right. \\
\nu=2\left\{\begin{matrix} \begin{matrix} \}b_1 \\ \end{matrix} & \begin{matrix} b_2 \\ b_4 \end{matrix} & \begin{matrix} \}b_3 \\ \end{matrix} & \left\{\begin{matrix} c_0 \\ c_2 \end{matrix}\right. & \left\{\begin{matrix} c_1 \\ c_3 \end{matrix}\right. & \quad & \left\{\begin{matrix} b_1 \\ b_3 \end{matrix}\right. & \left\{\begin{matrix} b_2 \\ b_4 \end{matrix}\right. & \begin{matrix} c_0 \\ c_2 \end{matrix} & \begin{matrix} \}c_1 \\ \end{matrix} & \begin{matrix} \}c_3 \\ \end{matrix} \end{matrix}\right. \\
\nu=1\left\{\begin{matrix} \begin{matrix} b_1 \\ b_4 \end{matrix} & \begin{matrix} b_2 \\ b_3 \end{matrix} & \begin{matrix} c_0, c_2 \\ c_1, c_3 \end{matrix} & \quad & \begin{matrix} b_1, b_3 \\ b_2, b_4 \end{matrix} & \begin{matrix} c_0 \\ c_1 \end{matrix} & \begin{matrix} c_2 \\ c_3 \end{matrix} \end{matrix}\right. \\
\nu=3\left\{\begin{matrix} \begin{matrix} b_1 \\ b_2 \end{matrix} & \begin{matrix} b_3 \\ b_4 \end{matrix} & \begin{matrix} c_0, c_2 \\ c_3, c_1 \end{matrix} & \quad & \begin{matrix} b_1, b_3 \\ b_4, b_2 \end{matrix} & \begin{matrix} c_0 \\ c_3 \end{matrix} & \begin{matrix} c_2 \\ c_1 \end{matrix} \end{matrix}\right.
\end{array}\right.
\end{array}$$

In order to identify these linear forms with certain configurations of points and O, E quadrics associated [cf. F. G. I, §§ 1, 3] with a linear complex or null system C_p in an odd space S_{2p-1}, the basis notation described in the next section is very convenient.

§ 2. *The Basis Notation for C_p. Certain Subgroups of the Group GC_p.*

1°. With $2p+2$ subscripts $1, 2, \ldots, 2p+2$, the $2^{2p}-1$ points of S_{2p-1} (mod. 2) are indicated by

$$P_{12},\ P_{1234},\ P_{123456},\ \ldots,\ \text{ where } P_{1,2,\ldots,2k}=P_{2k+1,\ldots,2p+2}.$$

Furthermore, $P_{1,2,\ldots,2p+2}$ represents no point, and among the subscripts of a point like subscripts cancel. The three points

$$P_{i_1i_2\ldots i_{2k}},\quad P_{j_1j_2\ldots j_{2l}},\quad P_{i_1i_2\ldots i_{2k}j_1j_2\ldots j_{2l}}$$

are on a line. By means of this condition all linear spaces in S_{2p-1} can be constructed [F. G. II, § 1].

2°. The points $P_{i_1i_2\ldots}$ and $P_{j_1j_2\ldots}$ are syzygetic or azygetic, according as they have an even or an odd number of common subscripts. This relation between the two points is mutual and a point is syzygetic to itself. The locus of points syzygetic to a given point is an S_{2p-2} on the given point, and is the null space of the given point in the null system C_p. Each null space may be named like its null point [F. G. II, Theo. 1].

3°. The 2^{2p} quadrics Q, whose polar system coincides with the null system C_p, divide into $2^{p-1}(2^p+1)E$ quadrics each containing $2^{p-1}(2^p+1)-1$ points and $2^{p-1}(2^p-1)O$ quadrics each containing $2^{p-1}(2^p-1)-1$ points [F. G. I, § 3].

4°. In the base notation the E quadrics are denoted by

$$Q_{1,2,\ldots,p+1-4j}=Q_{p+1-4j+1,\ldots,2p+2}\qquad (j=0,1,\ldots);$$

and the O quadrics are denoted by

$$Q_{1,2,\ldots,p-1-4j}=Q_{p-1-4j+1,\ldots,2p+2}\qquad (j=0,1,\ldots).$$

Thus if p is odd there is a quadric Q without subscripts, and this is an O or E quadric according as $p\equiv 1$ or 3, mod. 4. A pair of quadrics determines a point, and a quadric and a point determines another quadric, by virtue of the relation

$$Q_{i_1i_2\ldots}+Q_{j_1j_2\ldots}+P_{i_1i_2\ldots j_1j_2\ldots}=0\ \text{ [F. G. II, Theos. 4, 6].}$$

5°. A point lies on a quadric if half the number of subscripts in a symbol for the point together with the number of subscripts common to the symbols of the point and quadric is even [F. G. II, Theo. 7].

6°. The group GC_p of the null system has the order

$$NC_p=2^{p^2}(2^{2p}-1)(2^{2p-2}-1)\dots(2^2-1).$$

It is generated by a conjugate set of involutions $I_{i_1i_2\dots}$ each of which is associated with a $P_{i_1i_2\dots}$. The $I_{i_1i_2\dots}$ leaves every point syzygetic with $P_{i_1i_2\dots}$ unaltered, and sends every point P azygetic with $P_{i_1i_2\dots}$ into the point $P+P_{i_1i_2\dots}$ [F. G. I, § 1].

7°. The group of an O or E quadric has the order

$$NE_p=2^{p^2-p+1}(2^p-1)(2^{2p-2}-1)(2^{2p-4}-1)\dots(2^2-1);$$
$$NO_p=2^{p^2-p+1}(2^p+1)(2^{2p-2}-1)(2^{2p-4}-1)\dots(2^2-1).$$

The involution $I_{i_1i_2\dots}$ transforms a quadric Q into itself if $P_{i_1i_2\dots}$ is *not* on Q, otherwise it transforms Q into $Q+P_{i_1i_2\dots}$. The group of the quadric is generated by the involutions I attached to the points not on the quadric. This group is simply transitive on the set of points on the quadric and on the set of points not on the quadric. The points on Q added to Q furnish the remaining quadrics of the same type as Q; the points not on Q furnish the quadrics of the opposite type [F. G. I, §3, II, Theo. 7].

8°. The process of section and projection by a null space and from a null point is described in F. G. I, § 2, and used in § 5. So far as we shall need it here the process is as follows: To fix ideas let the point be P_{12}, its null space be L_{12}. Then the points on L_{12} apart from P_{12} are paired on null lines of the form

$$P_{12},\quad P_{i_1i_2\dots i_{2j}},\quad P_{12i_1i_2\dots i_{2j}}.$$

Projected from P_{12} these lines become the $2^{2\pi}-1$ points of an $S_{2\pi-1}$ ($\pi=p-1$) whose basis notation has subscripts 3, 4, ..., $2p+2$, and the above line corresponds to the point $P'_{i_1i_2\dots i_{2j}}$ in $S_{2\pi-1}$. The remaining $2^{2\pi}$ ordinary lines on P_{12} of the form

$$P_{12},\quad P_{1i_1i_2\dots i_{2j-1}},\quad P_{2i_1i_2\dots i_{2j-1}}$$

have no trace in $S_{2\pi-1}$. If points $P_{i_1i_2\dots}$, $P_{j_1j_2\dots}$ on L_{12} give rise to null lines on P_{12}, or points $P'_{i_1i_2\dots}$, $P'_{j_1j_2\dots}$ in $S_{2\pi-1}$, these points P' are syzygetic or azygetic according as the original points P are syzygetic or azygetic. Thus the null system C_p in S_{2p-1} determines its projection and section C'_π in $S_{2\pi-1}$.

9°. The 2^{2p-1} quadrics on P_{12} are

$$Q_{1i_1i_2\dots}=Q_{2j_1j_2\dots},$$

the sets of subscripts being complementary. These quadrics are paired under I_{12} into pairs $Q_{1i_1i_2\dots}$, $Q_{2i_1i_2\dots}$. The members of a pair have the same section by L_{12} and the null lines on P_{12} are either generators of or tangents to both quadrics. The points of $S_{2\pi-1}$ corresponding to a common generator or a common tangent of the pair is a point on or off respectively the quadric $Q'_{i_1i_2\ldots\ldots}$ in $S_{2\pi-1}$. In this way the 2^{2p-1} quadrics on P_{12} give rise to $2^{2\pi}$ pairs or to the $2^{2\pi}$ quadric Q' in $S_{2\pi-1}$ associated with C'_π. The quadric Q' is an E or O quadric according as the original pair is a pair of E or a pair of O quadrics.

We shall now apply the above notation to derive the generators, the order, and the structure of certain subgroups of GC_p. The first group which we shall consider is that group H_{12} which leaves the point P_{12} unaltered. There being $2^{2p}-1$ points P_{12} the order of H_{12} is

$$NH_{12}=NC_p\div(2^{2p}-1)=2^{2p-1}NC_\pi \qquad (\pi=p-1).$$

The null lines on P_{12} are permuted among themselves by H_{12} according to a group GC'_π of order NC_π. For if $P_{i_1i_2\ldots\ldots}$ or $P_{12i_1i_2\ldots\ldots}$ is any point on L_{12}, then either $I_{i_1i_2\ldots\ldots}$ or $I_{12i_1i_2\ldots\ldots}$ will permute the null lines on P_{12} just as the involution $I'_{i_1i_2\ldots\ldots}$ attached to the point $P'_{i_1i_2\ldots\ldots}$ of the derived space $S_{2\pi-1}$ will permute the points of this space. But these involutions generate the group GC'_π. On the other hand the elements of H_{12} must leave the derived null system C'_π unaltered. Hence, to account for the above order of H_{12}, we must determine the elements of H_{12}, which leave P_{12} and every null line on it unaltered.

We had noted that $I_{i_1i_2\ldots\ldots}$ and $I_{12i_1i_2\ldots\ldots}$ effect the same involution on the null lines on P_{12}. Since two involutions I are permutable if their points are syzygetic, we see that the involution

$$T_{i_1i_2\ldots\ldots}=I_{i_1i_2\ldots\ldots}I_{12i_1i_2\ldots\ldots}=I_{12i_1i_2\ldots\ldots}I_{i_1i_2\ldots\ldots}$$

effects the identical permutation of the null lines. The same is true of the involution

$$S_{i_1i_2\ldots\ldots}=I_{12}T_{i_1i_2\ldots\ldots}=T_{i_1i_2\ldots\ldots}I_{12}=I_{12}I_{i_1i_2\ldots\ldots}I_{12i_1i_2\ldots\ldots}.$$

The two involutions $S_{i_1i_2\ldots\ldots}$, $T_{i_1i_2\ldots\ldots}$ are associated with a null line on P_{12}, the former symmetrically and the latter with P_{12} isolated, and may therefore be named by the point $P'_{i_1i_2\ldots\ldots}$ of $S_{2\pi-1}$. We shall therefore speak of $S_{P'}$ and $T_{P'}$ where P' is any one of the $2^{2\pi}-1$ points of $S_{2\pi-1}$. The $2(2^{2\pi}-1)$ elements $T_{P'}$, $S_{P'}$ and the two elements 1, I_{12} constitute the group $\overline{H}_{12}$, which leaves P_{12} and every null line on it unaltered.

The group $\overline{H}_{12}$ is Abelian with involutory elements and its multiplication table is:

$$I_{12}S_{P'}=T_{P'}, \quad I_{12}T_{P'}=S_{P'};$$

if P', $\overline{P}'$ are syzygetic, then

$$T_{P'}T_{\overline{P}'}=S_{P'+\overline{P}'}, \quad T_{P'}S_{\overline{P}'}=T_{P'+\overline{P}'}, \quad S_{P'}S_{\overline{P}'}=S_{P'+\overline{P}'};$$

if P', $\overline{P}'$ are azygetic, then

$$T_{P'}T_{\overline{P}'}=T_{P'+\overline{P}'}, \quad T_{P'}S_{\overline{P}'}=S_{P'+\overline{P}'}, \quad S_{P'}S_{\overline{P}'}=T_{P'+\overline{P}'}.$$

The first two of these relations follow directly from the definitions of S, T; the first in each of the next sets is deduced by verifying the effect of the two members upon the $2p-1$ linearly independent points $P_{12}, P_{13}, \ldots, P_{1,2p+2}$; the remaining ones follow directly from these by inserting factors I_{12}.

The elements $T_{P'}$, $S_{P'}$ of $\overline{H}_{12}$ will interchange or leave unaltered the points on a null line of P_{12} which corresponds to $\overline{P}'$ in $S_{2\pi-1}$ according as P' and $\overline{P}'$ are azygetic or syzygetic. We ask for elements of $\overline{H}_{12}$ which effect the identical collineation in L_{12}. Such an element would leave unaltered all the S_{2p-2}'s on P_{12}, since these are null spaces of points on L_{12}, and therefore *all* the lines on P_{12}. If then it sends the point $P_{1i_1i_2\ldots.}$ outside of L_{12} into the point $P_{2i_1i_2\ldots.}$ it must interchange the null spaces of these points, and therefore interchange the points in which any ordinary line on P_{12} meets these spaces. Thus 1, I_{12} are the only elements of $\overline{H}_{12}$ which leave every point on L_{12} unaltered.

The group H_{12} contains subgroups simply isomorphic with the GC'_{π} and in fact a conjugate set of $2^{2\pi}$ such subgroups. Let $P_{1i_1i_2\ldots.}$, $P_{2i_1i_2\ldots.}$ be the pair of points on any one of the $2^{2\pi}$ ordinary lines on P_{12}. Their null spaces cut L_{12} in the same $S_{2\pi-1}$ which meets every null line on P_{12} in a single point $\overline{P}$. The $2^{2\pi}-1$ involutions $I_{\overline{P}}$ generate a group isomorphic with the group GC'_{π} on the null lines. Moreover, the group thus generated can contain no element of $\overline{H}_{12}$ since the order of the group of P_{12} and $P_{1i_1i_2\ldots.}$ is $NH_{12}\div 2^{2p-1}=NC'_{\pi}$. A group of this kind occurs for each one of the conjugate set of $2^{2\pi}$ ordinary lines. Incidentally we have shown that

(15) *The group which leaves two azygetic points in S_{2p-1} each unaltered, has the order NC_{π}, and is generated by the involutions I attached to all points syzygetic with the two given points.*

No element $T_{P'}$ or $S_{P'}$ can leave an ordinary line on P_{12} unaltered. For if it did the one or the other of the two would leave each of the two remaining points of the line unaltered, and the group of one of these points and P_{12} would have an element in common with $\overline{H}_{12}$ contrary to what has been proved above. Hence

(16) *The subgroup* H_{12} *of* GC_p, *which leaves a point* P_{12} *and its null space* L_{12} *unaltered has the order* $2\,.\,2^{2\pi}.\,NC'_{\pi}$ $(\pi=p-1)$ *and is generated by the involutions* I *attached to points syzygetic with* P_{12}. *It has an invariant subgroup* $\bar{H}_{12}$ *of order* $2\,.\,2^{2\pi}$ *which leaves every null line on* P_{12} *unaltered. The factor group of* $\bar{H}_{12}$ *under* H_{12} *is a* GC_{π}. *Also* $\bar{H}_{12}$, *an Abelian group, has an invariant* g_2 *which leaves every point on* L_{12} *unaltered. The factor group of* g_2 *under* $\bar{H}_{12}$ *is a regular group on the* $2^{2\pi}$ *ordinary lines on* P_{12}. *The group* H_{12} *has a set of* $2^{2\pi}$ *conjugate subgroups* GC_{π} *and is the direct product of* $\bar{H}_{12}$ *and any one of these subgroups.*

Let Q be a quadric on P_{12}. Its section by L_{12} and projection from P_{12} is a quadric Q' in $S_{2\pi-1}$ of the same kind. The points of Q' in $S_{2\pi-1}$ arise from the generators of Q on P_{12}; the points not on Q' arise from the tangents of Q on P_{12}. The group which leaves Q unaltered has the order $NC_p \div 2^{p-1}(2^p \pm 1)$, the upper or lower sign being used according as Q is an E or O quadric. Since P_{12} is any one of the set of $2^{p-1}(2^p\pm 1)-1$ conjugate points on Q, the order of the group which leaves Q and P_{12} unaltered is $2^{2\pi}NQ'_{\pi}$. Clearly Q and P_{12} are unaltered by involutions I attached to all points on L_{12} which are not on Q. Moreover, in $S_{2\pi-1}$ these generate the GQ'_{π}. Hence the group of Q, P_{12} must contain an invariant subgroup of order $2^{2\pi}$ which consists of elements $T_{P'}$ or $S_{P'}$. If $P_{i_1i_2\ldots\ldots}$ is not on Q, then $P_{12i_1i_2\ldots\ldots}$ is not on Q and $T_{i_1i_2\ldots\ldots}$ leaves Q, P_{12} unaltered. If $P_{i_1i_2\ldots\ldots}$ is on Q, $P_{12i_1i_2\ldots\ldots}$ also is on Q, and $S_{i_1i_2\ldots\ldots}$ leaves Q, P_{12} unaltered. That the group of P_{12}, Q is generated by the involutions I attached to points on L_{12}, but not on Q, follows first from the fact proved above that they generate the factor group GQ'_{π}. Secondly they evidently generate the elements $T_{P'}$ where P' is a point of $S_{2\pi-1}$ not on Q' and, since among these points P' are 2π linearly independent points, the products of these must according to the multiplication table above, give rise to an element $T_{P'}$ or $S_{P'}$ for every point P'. But as we have seen, only elements $S_{P'}$ are admissible when P' is on Q'. Now the group of Q, P_{12} is also the group of Q, $\bar{Q}$ where $\bar{Q}=Q+P_{12}$, whence

(17) *The subgroup of* GC_p *which leaves two quadrics* Q, $\bar{Q}$ *of the same type each unaltered has the order* $2^{2\pi}NQ'_{\pi}$, *and is generated by the involutions* I *attached to all points not on either of the two quadrics. It has an invariant Abelian subgroup which consists of the involutions* $T_{P'}$ *or* $S_{P'}$ *according as* P' *is not or is a point of the projected quadric* Q' *in* $S_{2\pi-1}$. *The factor group is simply isomorphic with the group of* Q'.

We see also from what has been said above that

(18) *The group* $\bar{H}_{12}$, *which leaves all null lines on* P_{12} *unaltered, has a subgroup of order* $2^{2\pi}$ *corresponding to each pair* Q, $\bar{Q}$ *of quadrics on* P_{12}. *This subgroup is a regular group on the ordinary lines on* P_{12}.

Two quadrics Q, $\bar{Q}$ of different kinds, such that $Q+\bar{Q}=P_{12}$, meet in their common section by L_{12}. Each is unaltered by the involution I attached to any point not on either. But all such points are on L_{12}. Any null line on P_{12} (which is not on either quadric) touches the quadrics at a common point and contains therefore a point not on either one. Hence these involutions will generate the group GC'_π on the null lines of P_{12}. Moreover, I_{12} will leave every null line and also Q, $\bar{Q}$ unaltered. Hence the order of the group of Q, $\bar{Q}$ is at least $2NC'_\pi$. The group of Q has the order $NC_p \div 2^{p-1}(2^p \pm 1)$ and it is transitive on the quadrics $\bar{Q}$ of the other kind, whence the order of the group of Q, $\bar{Q}$ is

$$NC_p \div \{2^{p-1}(2^p \pm 1)\}\{2^{p-1}(2^p \pm 1)\} = 2NC'_\pi .$$

Hence

(19) *The subgroup of* GC_p, *which leaves two quadrics* Q, $\bar{Q}$ *of opposite types, such that* $Q+\bar{Q}=P_{12}$ *unaltered, has the order* $2NC'_\pi$ *and is generated by the involutions* I *attached to all points not on either quadric. It has an invariant* g_2, *namely* 1, I_{12}, *whose factor group is simply isomorphic with* GC'_π.

We wish now to determine the order and nature of the subgroup $[I]$ generated by involutions $[I]$ attached to all points syzygetic with a pair of syzygetic points P_{12}, P_{34}, and therefore to P_{1234} also, excluding the generators I_{12}, I_{34}, I_{1234}. We note first from the multiplication table above that

$$I_{12} = I_{56} I_{1256} I_{78} I_{1278} I_{5678} I_{125678} ,$$

so that the elements excluded as generators appear in the group $[I]$. The group is an invariant subgroup of the group $H_{12,34}$, the subgroup of GC_p which leaves P_{12}, P_{34} each unaltered. The order of $H_{12,34}$ is

$$2^{2p-1} NC_\pi \div 2(2^{2\pi}-1) = 2^{4p-5} NC_{\pi'} \qquad (\pi'=\pi-1),$$

since the $2(2^{2\pi}-1)$ points P_{34} syzygetic with P_{12} are conjugate under H_{12}. If we apply Theorem (16) to the points P' of $S_{2\pi-1}$ and determine thereby the group H'_{34} in $S_{2\pi-1}$ we find that the involutions $[I]$ generate a factor group of permutations of the points P', or null lines on P_{12}, whose order is $2^{2\pi-1} NC_{\pi'}$. This factor group corresponds to the invariant subgroup of the group $[I]$

made up of all elements $T_{P'}$ or $S_{P'}$ of $\bar{H}_{12}$ which leave P_{34} unaltered and which are generated by $[I]$. The pair $T_{P'}$, $S_{P'}$ leave P_{34} unaltered only if P' is one of the $(2^{2p-4}-1)-1$ points distinct from P'_{34} and syzygetic with P'_{34}. These pairs are generated by $[I]$ as are also the four additional elements $1, I_{12}$, $I_{34}I_{1234}$, $I_{12}I_{34}I_{1234}$, whence the required invariant subgroup is of order 2^{2p-3}, and the order of the group $[I]$ is $2^{4p-6}NC_{\pi'}$ and its index under $H_{12,34}$ is 2. Hence

(20) *The group generated by the involutions $[I]$ attached to all points syzygetic with two given syzygetic points, but not on the null line of the two points has the order $2^{4p-6}NC_{\pi'}$ $(\pi'=p-2)$. It has an invariant subgroup of order 2^{4p-6}, whose factor group is $GC_{\pi'}$, which effects the identical permutation on the null planes through the line on the given points. This invariant subgroup has an invariant subgroup of order 2^{2p-3} which effects the identical permutation on the null lines through one of the given points. The group $[I]$ itself is an invariant subgroup of index 2 under the group of the two points.*

Given four quadrics Q, Q', Q'', Q''' of the same type such that $Q+Q'=P_{12}$, $Q''+Q'''=P_{12}$, $Q+Q''=P_{34}$, $Q'+Q'''=P_{34}$; we ask for the order of the group generated by the involutions $[J]$ attached to points on none of the quadrics. In $S_{2\pi-1}$ the pairs Q, Q' and Q'', Q''' become quadrics R, R'' where $R+R''=P'_{34}$, and according to Theorem (17) the group generated by $[J]$ has a factor group of order $2^{2\pi'}NQ_{\pi'}$ which corresponds to an invariant subgroup of $[J]$ consisting of those elements of $\bar{H}_{12}$ which leave each of the four quadrics unaltered. These are the identity, all elements $T_{P'}$ for which P' is not on either R, R'', and all elements $S_{P'}$ for which P' is on both R' and R''. Such points P' are all points in $S_{2\pi-1}$ syzygetic with P'_{34} including P'_{34}, whence the group has the order $2^{2\pi-1}$. Hence

(21) *The group generated by the involutions $[I]$ attached to all points not on the four quadrics which arise from any one by transforming it by involutions I attached to three points on one of its generators is of order $2^{4p-7}NQ_{\pi'}$ $(\pi'=p-2)$, where $Q_{\pi'}$ is of the same type as the given quadrics. It has an invariant subgroup of order 2^{4p-7} whose factor group is isomorphic with $GQ_{\pi'}$.*

If, on the other hand, Q'', Q''' are of a type opposite to that of Q, Q' then in $S_{2\pi-1}$, R, R'' are unlike quadrics, and we apply Theorem (19) to find the group of order $2NC_{\pi'}$ in $S_{2\pi-1}$. Here the only elements of $\bar{H}_{12}$ are 1, $I_{34}I_{1234}$.

(22) *If* $\overline{P_{12}, P_{34}}$ *is a tangent to* Q *at* P_{12}, *the group generated by involutions* I *attached to all points not on the four quadrics* Q, $Q+P_{12}$, $Q+P_{34}$, $Q+P_{1234}$ *is of order* $2^2NC_{\pi'}$ $(\pi'=p-2)$. *It has an invariant subgroup made up of* 1, I_{34}, I_{1234}, $I_{34}I_{1234}$ *whose factor group is simply isomorphic with* $GC_{\pi'}$.

The groups described above include all which appear in the next section where the sixteen cases of $g^{(2)}_{n,k}$ are discussed.

§ 3. *Identification of the Group* $g^{(2)}_{n,k}$.

This identification will be effected by comparing the permutations of the forms b, c under the operations of $g^{(2)}_{n,k}$ which is generated by the transpositions such as T_{12} and by $A_{1,\ldots,k+1}$ with the permutations of certain sets of points or quadrics in the finite geometry of the null system under involutions I attached to certain points. In order that the isomorphism may be one-to-one it is necessary that the geometric objects in the conjugate set which correspond to $B_{1,\ldots,}B_n$ and C_0 be explicitly given; and secondly, that the conjugate set of forms b_2 be explicitly attached to a conjugate set of points. The first requirement ensures that under a given element the transforms of $x_1 \ldots, x_n$, x_0 can be uniquely determined; the second requirement ensures that the conjugate set of generating involutions can be determined since the transposition T_{12} has the invariant linear form B_{12}. So far as the other forms are concerned it is not necessary to identify them, but we shall usually do this sometimes explicitly, sometimes only so far as they occur in pairs.

In order to illustrate the easy passage from the permutation of the points to the algebraic transformation some invariant subgroups of $g^{(2)}_{n,k}$ are derived. The same division as in § 1 into sixteen cases is made, and in each of these the final description of the $g^{(2)}_{n,k}$ is given in the table (30) at the end of this section where also a reference to § 2 for a more complete description of the $g^{(2)}_{n,k}$ is made. The notation for the geometry is that of § 2 with subscripts 1, 2, $\ldots$, n with some additional subscripts selected from α, β, γ, δ.

Throughout the

CASE I: k even

we shall identify the generators as follows:

$$T_{12}=I_{12}, \quad A_{1,\ldots,k+1}=I_{1,\ldots,k+1,\alpha}.$$

For the cases $\nu=0, 1, 2, 3$ coming under $\varkappa=0$ we identify the conjugate sets of linear forms with the following sets of points in the finite geometry:

$$(23)\quad \varkappa=0 \left\{ \begin{array}{l} \nu=0, 2: n+2 \text{ subscripts } 1, \ldots, n, \alpha, \beta. \quad b_2=P_{12},\ c_1=P_{1\alpha}; \\ b_1=P_{1\beta},\ b_0=P_{\alpha\beta};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta};\quad b_4=P_{1234},\ c_3=P_{123\alpha}. \\ \nu=1, 3: n+3 \text{ subscripts } 1, \ldots, n, \alpha, \beta, \gamma. \quad b_2=P_{12},\ c_1=P_{1\alpha}; \\ b_1=P_{1\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_3=P_{123\alpha};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta}. \end{array} \right.$$

One difference between these cases is noteworthy. When $\nu=0, 2$ paired forms are represented by the same sets of points, but this is not serious, for we need only to identify completely the conjugate sets b_2, c_1 and b_1, c_0 in order to identify completely the conjugate set of generators, and the conjugates of the reference forms. But when $\nu=1$, the forms b_1, c_0 are paired, and when $\nu=3$ the forms b_2, c_1 are paired, and in this case the forms of a pair must be separated, as they are in fact by the above notation. It is to be understood, of course, that the notation $b_2=P_{12}$ indicates

$$B_{1,2}=P_{12},\quad B_{1,3}=P_{13},\quad B_{2,3}=P_{23}, \text{ etc.};$$
$$B_{1,2,3,4,5,6}=P_{123456}, \text{ etc.};\quad B_{1,2,\ldots,10}=P_{12\ldots10};\quad \ldots\ldots$$

We have next to verify that in each of the four cases the generators T_{12}, $A_{1,\ldots,k+1}$ permute the forms just as the corresponding involutions I_{12}, $I_{1,\ldots,k+1,\alpha}$ permute the corresponding points. The rule for permuting the forms is given in § 1 (9), that for permuting the points in § 6, 6°, 2°. The consideration of a few typical instances which will be omitted here shows that the two sets of permutations are isomorphic.

The generating involutions are the involutions I attached to all points of the sets P_{12}, $P_{1,2,\ldots,k+1,\alpha}$. The various cases now divide as follows:

$\nu=0$. The generators are attached to all points not on the quadric Q_β. Here $2p+2=n+2=4\mu+2$, $p+1=2\mu+1$ and Q_β is an E or O quadric in S_{n-1} according as $\mu\equiv0, 1$, mod. 2, or $n\equiv0, 4$, mod. 8.

$\nu=2$. The generators I are attached to all points not on the quadric Q. Here $2p+2=4\mu+4$, $p+1=2\mu+2$, and Q is an O or E quadric in S_{n-1} according as $\mu\equiv0, 1$, mod. 2, or $n\equiv2, 6$, mod. 8.

$\nu=1$. The generators are attached to all points not on the quadrics Q, $Q_{\beta\gamma}$ of opposite type. Here $2p+2=4\mu+4$, $2\pi-1=4\mu-1=n-2$.

$\nu=3$. The generators are attached to all points not on the like quadrics Q_β, Q_γ. Here $2p+2=4\mu+6$, $2\pi-1=4\mu+1=n-2$.

The salient features of the group in each of these four cases are tabulated in (30) at the end of this section, and in each case a reference to § 2 is made for a more complete description of the group.

The four cases which come under $\varkappa=2$ do not differ materially from the four just considered, and it will be sufficient to set forth for them a table analogous to (23) and to indicate in each case the position of the generators I. The details for each case can then be supplied from table (30).

$$(24)\quad \varkappa=2\left\{\begin{array}{l} \nu=0,\ 2: n+2 \text{ subscripts } 1,\ \ldots,\ n,\ \alpha,\ \beta.\quad b_2=P_{12},\ c_3=P_{123\alpha};\\ b_1=P_{1\beta},\ c_2=P_{12\alpha\beta};\quad b_3=P_{123\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_1=P_{1\alpha}.\\ \nu=1,\ 3: n+3 \text{ subscripts } 1,\ \ldots,\ n,\ \alpha,\ \beta,\ \gamma.\quad b_2=P_{12},\ c_3=P_{123\alpha};\\ b_1=P_{1\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_1=P_{1\alpha};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta}. \end{array}\right.$$

The generators I are attached to all points which,

when $\nu=0$, are not on the quadric Q_α;
when $\nu=2$, are not on the quadric $Q_{\alpha\beta}$;
when $\nu=1$, are not on the like quadrics $Q_{\alpha\beta}$, $Q_{\alpha\gamma}$;
when $\nu=3$, are not on the unlike quadrics Q_α, $Q_{\alpha\beta\gamma}$.

A reference to the table shows that when $\nu=3$, $g_{n,k}^{(2)}$ has an invariant g_2; when $\nu=1$, an invariant $g_{2^{n-1}}$. As an example of the transition from the collineations in the finite geometry to the linear transformations of the variables $x_0,\ \ldots,\ x_n$ we shall derive the equations of these invariant subgroups beginning with $\nu=3$.

According to Theorem (19) the invariant g_2 is the involution $I_{\beta\gamma}$. Under $I_{\beta\gamma}$ the point $P_{1\beta}$ becomes $P_{1\gamma}=P_{2,\ldots,n,\alpha,\beta}$, and $P_{\alpha\beta}$ becomes $P_{\alpha\gamma}=P_{1,\ldots,n,\beta}$. Hence the form B_1 becomes $c_{2,3,\ldots,n}$, and the form C_0 becomes $B_{1,\ldots,n}$. But this is precisely the effect of the linear transformation

$$x_i'=x_i+L\quad (i=0,\ 1,\ \ldots,\ n;\ L=x_0+x_1+\ldots+x_n).$$

When $\nu=1$, the null lines on $P_{\beta\gamma}$, the point common to the two quadrics, are either

$$P_{\beta\gamma},\ P_{12},\ P_{3,\ldots,n,\alpha}\quad\text{or}\quad P_{\beta\gamma},\ P_{1234},\ P_{5,\ldots,n,\alpha}.$$

On taking a section and projection from $P_{\beta\gamma}$ the subscripts β, γ are dropped and the pair of quadrics $Q_{\alpha\beta}$, $Q_{\alpha\gamma}$ becomes Q_α' in $S_{2\pi-1}$, while the null lines are either points P_{12}' or points P_{1234}' according as the number of subscripts $\not\equiv 0$ or $\equiv 0$, mod. 4. Points P_{12}' are not on Q_α', points P_{1234}' are on Q_α'. According to Theorem (17) the elements of the invariant subgroup are either

$$I_{12}I_{3,\ldots,n,\alpha}\quad\text{or}\quad I_{1234}I_{5,\ldots,n,\alpha}I_{\beta\gamma}.$$

Taking the first as a sample of its type we find that it transforms $P_{\alpha\beta}$, $P_{1\beta}$, $P_{3\beta}$ into $P_{3,\ldots\ldots,n,\beta}$, $P_{2\beta}$, $P_{4,\ldots\ldots,n,\alpha,\beta}$, and therefore sends C_0, B_1, B_3 into $B_{3,\ldots\ldots,n}$, B_2, $C_{4,\ldots\ldots,n}$ respectively, whence it is

$$x_0'=x_0+C_{3,\ldots\ldots,n},\quad x_i'=x_i+B_{1,2},\quad x_j'=x_j+C_{3,\ldots\ldots,n}\qquad (i=1,2;\ j=3,\ldots\ldots,n).$$

The general element of this type is obtained from this by shifting sets of four subscripts from the form C to the form B and is

$$x_0'=x_0+C,\quad x_i'=x_i+B,\quad x_j'=x_j+C$$
$$(i=1,\ \ldots\ldots,\ 4c+2;\ j=4c+3,\ \ldots\ldots,\ n)$$
$$B=B_{1,2,\ldots\ldots,4c+2},\quad C=C_{4c+3,\ldots\ldots,n}.$$

Similarly we find that the general element of the other type is

$$x_0'=x_0+B,\quad x_i'=x_i+C,\quad x_j'=x_j+B$$
$$(i=1,\ \ldots\ldots,\ 4c;\ j=4c+1,\ \ldots\ldots,\ n)$$
$$B=B_{1,\ldots\ldots,4c},\quad C=C_{4c+1,\ldots\ldots,n}.$$

We take up now the

CASE II: k odd.

There is a sharp difference between the cases $\varkappa=1$ and $\varkappa=3$. When k is odd the generators § 1 (4) generate a group $g_{n,k}'^{(2)}$ on the variables $x_1,\ldots\ldots,x_n$ alone, and $g_{n,k}'^{(2)}$ is necessarily isomorphic with $g_{n,k}^{(2)}$. In the case $\varkappa=3$ there is according to § 1 (7), an invariant quadratic form which contains the variable x_0, and from the invariance of this form the effect of any element of $g_{n,k}^{(2)}$ upon x_0 can be found when its effect upon the variables $x_1,\ldots\ldots,x_n$ is known. Hence in this case the group $g_{n,k}'^{(2)}$ is simply isomorphic with $g_{n,k}^{(2)}$. However, in the case $\varkappa=1$ there may be an invariant subgroup of order μ of $g_{n,k}^{(2)}$ whose elements have the form

$$x_0'=x_0+F_j,\quad x_i'=x_i\qquad (i=1,\ \ldots\ldots,\ n;\ j=1,\ \ldots\ldots,\ \mu),$$

and then $g_{n,k}^{(2)}$ is in $\mu-1$ isomorphism with $g_{n,k}'^{(2)}$.

We shall therefore consider these cases separately and begin with

CASE II 1: $\varkappa=1$,

and determine first the group $g_{n,k}'^{(2)}$. According to table (14) we have now four conjugate sets of forms b_1, b_2, b_3, b_4, and the sets b_1, b_2 must be completely identified in order to determine respectively the linear transformations and the conjugate generators. We shall take for

$\nu=0, 2$: subscripts $1, 2, \ldots\ldots, n, \alpha, \beta$,
$\nu=1, 3$: subscripts $1, 2, \ldots\ldots, n, \alpha$.

We identify the generators and forms b as follows:

(25) $T_{12}=I_{12}$, $A_{1,\ldots\ldots,k+1}=I_{1,\ldots\ldots,k+1}$; $b_2=P_{12}$; $b_1=P_{1\alpha}$; $b_3=P_{123\alpha}$; $b_4=P_{1234}$.

Then from § 1 (12) we find that these linear forms are permuted by the generators precisely as the points are permuted by the involutions I, and we have only to identify the location of the generators.

$\nu=0$. The generators belong to all points not on the like quadrics Q_α, Q_β. Here $2p+2=4\mu+2$, $p+1=2\mu+1$, $2\pi-1=n-3$; and Q_α, Q_β are E or O quadrics according as $\mu\equiv0$, 1, mod. 2.

$\nu=2$. The generators belong to all points not on the unlike quadrics Q, $Q_{\alpha\beta}$. Here $2p+2=4\mu+4$, $2\pi-1=n-3$.

$\nu=1$. The generators belong to all points not on Q_α. Here $2p+2=4\mu+2$, $p+1=2\mu+1$, and Q_α is an E or O quadric according as $\mu\equiv0$, 1, mod. 2.

$\nu=3$. The generators belong to all points not on Q. Here $2p+2=4\mu+4$, $p+1=2\mu+2$, and Q is an E or O quadric according as $\mu\equiv1$, 0, mod. 2.

These facts concerning $g'^{(2)}_{n,k}$ are collected in the table:

(26)

$g'^{(2)}_{n,k} : \varkappa=1$

ν	Order of Invariant Subgroup	Factor Group	Invariant Quadric $n\equiv\nu'$, mod. 8	Reference
0	2^{n-2}	GQ_{n-3}	$Q={E \atop O}$ if $\nu'={0 \atop 4}$	§ 2 (17)
2	2	GC_{n-3}		§ 2 (19)
1		GQ_{n-2}	$Q={E \atop O}$ if $\nu'={1 \atop 5}$	§ 2, 7°
3		GQ_{n-2}	$Q={E \atop O}$ if $\nu'={7 \atop 3}$	§ 2, 7°

In order to obtain a representation which will separate the forms C, we will use when

$\nu=0, 2$: subscripts $1, 2, \ldots., n, \alpha, \beta, \gamma, \delta$;
$\nu=1, 3$: subscripts $1, 2, \ldots., n, \alpha, \beta, \gamma$;

and in all four of these cases will identify the generators and the forms C as follows:

(27) $T_{12}=I_{12}$, $A_{1,2,\ldots.,k+1}=I_{1,2,\ldots.,k+1,\alpha,\beta}$; $c_1=P_{1\alpha}$; $c_3=P_{123\beta}$; $c_0=P_{\alpha\gamma}$; $c_2=P_{12\beta\gamma}$.

Then the generators affect the forms c as is required in § 1 (9). Moreover, the behavior of the forms b can be deduced from that of the forms c for $B_1=C_0+C_1$. Taking up the four cases in order we find that when:

$\nu=0$. The generators belong to all points not on any one of the four like quadrics $Q_{\alpha\gamma}$, $Q_{\alpha\delta}$, $Q_{\beta\gamma}$, $Q_{\beta\delta}$, and the group is described in § 2 (21). Here $2p+2=4\mu+4=n+4$, $p+1=2\mu+2$ and the quadrics are of type E or O, according as $\mu\equiv 0, 1, \bmod. 2$.

$\nu=2$. The generators belong to all points not on any of the pairs of unlike quadrics Q_α, Q_β, $Q_{\alpha\gamma\delta}$, $Q_{\beta\gamma\delta}$. Here $2p+2=4\mu+6=n+4$.

$\nu=1$. The generators belong to all points not on the pair of like quadrics $Q_{\alpha\gamma}$, $Q_{\beta\gamma}$. Here $2p+2=4\mu+4=n+3$, $p+1=2\mu+2$ and the quadrics are E or O quadrics according as $\mu\equiv 0, 1, \bmod. 2$.

$\nu=3$. The generators belong to all points not on the like quadrics Q_α, Q_β. Here $2p+2=4\mu+6=n+3$, $p+1=2\mu+3$ and the quadrics are of type O, E, according as $\mu\equiv 0, 1$.

Thus a comparison of these results as listed in table (30) with the table (26) above, shows that $g_{n,k}^{(2)}$ has an invariant G_μ whose factor group is $g_{n,k}^{\prime(2)}$, and that $\mu=2^{n-1}$ except in the case $\nu=2$ for which $\mu=2$. When $\mu=2^{n-1}$ the subgroup G_μ must consist of elements of the form

$$x_0'=x_0+F,\ x_i'=x_i\quad (i=1,\ldots\ldots,n)\quad \text{where } F=x_{i_1}+x_{i_2}+\ldots\ldots+x_{i_{2j}}\quad (0 \overline{<} 2j \overline{<} n).$$

When $\mu=2$ G_μ consists of the identity and the single element $F=L$ the invariant linear form.

We take up finally the

CASE II 2 : $\varkappa=3$.

If in this case the forms c are completely identified, the transformations are determined. For all values of ν we take

$$(28)\qquad T_{12}=I_{12},\ A_{1,\ldots\ldots,k+1}=I_{1,\ldots\ldots,k+1};\ b_2=P_{12},\ b_4=P_{1234}.$$

Then for various values of ν we represent the forms c as follows:

(29)

ν	Subscripts 1, ... , n and	c_0	c_1	c_2	c_3
0	α, β	Q_α	Q_1	$Q_{12\alpha}$	Q_{123}
2	α, β	$Q_{\alpha\beta}$	$Q_{1\alpha}$	$Q_{12\alpha\beta}$	$Q_{123\alpha}$
1	α, β, γ	$Q_{\beta\gamma}$	$Q_{1\beta}$	$Q_{12\beta\gamma}$	$Q_{123\beta}$
3	α, β, γ	Q_α	$Q_{1\alpha\gamma}$	$Q_{12\alpha}$	$Q_{123\alpha\gamma}$

$\nu=0, 2$. The generators belong to all points syzygetic with $P_{\alpha\beta}$. Here $2p+2=4\mu+2=n+2$ and $S_{2\pi-1}=S_{n-3}$.

$\nu=1, 3$. The generators belong to all points syzygetic with the azygetic triad $P_{\beta\gamma}$, $P_{\gamma\alpha}$, $P_{\alpha\beta}$. Here $2p+2=4\mu+4=n+3$ and $S_{2\pi-1}=S_{n-2}$.

This completes the discussion of the sixteen cases, and the structure and orders of the various types of $g^{(2)}_{n,k}$ are readily ascertained from the table (30).

(30)

	ν	Order of Invariant Subgroup	Factor Group*	Invariant Quadric $n\equiv\nu'$, mod. 8	Reference to § 2
$\kappa=0$	0		GQ_{n-1}	$Q=\substack{E\\O}$ if $\nu'=\substack{0\\4}$	7°
	1	2	GC_{n-2}		(19)
	2		GQ_{n-1}	$Q=\substack{E\\O}$ if $\nu'=\substack{6\\2}$	7°
	3	2^{n-1}	GQ_{n-2}	$Q=\substack{E\\O}$ if $\nu'=\substack{7\\3}$	(17)
$\kappa=1$	0	$2^{n-2}.\,2^{n-1}$	GQ_{n-3}	$Q=\substack{E\\O}$ if $\nu'=\substack{0\\4}$	(21)
	1	2^{n-1}	GQ_{n-2}	$Q=\substack{E\\O}$ if $\nu'=\substack{1\\5}$	(17)
	2	$2\,.\,2$	GC_{n-3}		(22)
	3	2^{n-1}	GQ_{n-2}	$Q=\substack{E\\O}$ if $\nu'=\substack{7\\3}$	(17)
$\kappa=2$	0		GQ_{n-1}	$Q=\substack{E\\O}$ if $\nu'=\substack{0\\4}$	7°
	1	2^{n-1}	GQ_{n-2}	$Q=\substack{E\\O}$ if $\nu'=\substack{1\\5}$	(17)
	2		GQ_{n-1}	$Q=\substack{E\\O}$ if $\nu'=\substack{2\\6}$	7°
	3	2	GC_{n-2}		(19)
$\kappa=3$	0	2^{n-1}	GC_{n-3}		(16)
	1		GC_{n-2}		6°, (15)
	2	2^{n-1}	GC_{n-3}		(16)
	3		GC_{n-2}		6°, (15)

[In this table the subscript of Q and C is the dimension of the space in which the quadric or null system lies.]

It is proved in P. S. II (27) that the groups $g_{n,k}$ and $g_{n,n-k-2}$ are simply isomorphic and a transformation T which sends the one group into the other is given there. This transformation is degenerate modulo 2, so that it is not necessarily true that the modular groups $g^{(2)}_{n,k}$ and $g^{(2)}_{n,n-k-2}$ are simply isomorphic. We should expect, however, that some sort of isomorphism persists, and this is found to be the case. In the table below the distinct associated cases are opposite each other. We find from (30) that either the associated groups are simply isomorphic, or one has an invariant subgroup (noted on the margin) whose factor group is simply isomorphic with the other.

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* These factor groups are discussed in full in Dickson's "Linear Groups," Chapters II, VIII.

		$\varkappa, \nu$	$\varkappa' = \nu - \varkappa - 2, \nu$	
		0, 0	2, 0	
	g_2	0, 1	3, 1	
(31)		0, 3	1, 3	
		1, 1	2, 1	
		1, 2	3, 2	$g_{2^{n-3}}$
	g_2	2, 3	3, 3	

We are above all interested in the sets P^p_{2p+2}, which may be required to be self-associated. The four cases in question here abstracted from (30) are:

	p mod. 4	$2p+2$ mod. 8	Invariant Subgroup	Factor Group
	0	2		GO_{2p+1}
(32)	1	4	$G_2 4p+1$	GO_{2p-1}
	2	6		GO_{2p+1}
	3	0	$G_2 2p+1$	GC_{2p-1}

We shall show in the next section that the requirement that the set P^p_{2p+2} be self-associated does not reduce the group $g^{(2)}_{2p+2,\,p}$ when p is odd; but that when p is even this requirement reduces the group to one with an invariant G_2 whose factor group is GC_{2p-1}. Thus in all cases the self-associated set defines a modular group in the finite geometry of S_{2p-1}. In order to impose the conditions for self-association most conveniently, it is necessary to consider the ultra-elliptic set P^p_{2p+2}.

§ 4. *Modular Groups Determined by $e_{n,k}$.*

Congruent ultra-elliptic sets P^k_n, P'^k_n lie on projectively equivalent norm-elliptic curves E^{k+1}, E'^{k+1}. When E'^{k+1} is projected upon E^{k+1}, P'^k_n is projected upon a set on E^{k+1} whose elliptic parameters $u'_1, \ldots, u'_n$ are expressed in terms of the elliptic parameters $u_1, \ldots, u_n$ of P^k_n by means of a linear transformation—an element of the group $e_{n,k}$. The generators of $e_{n,k}$ are the transpositions of the u's and the element $A_{1,\ldots,k+1}$ which is (P. S. II (32))

$$A_{1,\ldots,k+1}: \begin{array}{ll} u'_i = u_i - \dfrac{2}{k+1}(u_1 + \ldots + u_{k+1}) & (i = 1, \ldots, k+1), \\[2ex] u'_j = u_j + \dfrac{k-1}{k+1}(u_1 + \ldots + u_{k+1}) & (j = k+2, \ldots, n). \end{array}$$

All the elements of $e_{n,k}$ have for coefficients rational numbers with denominators which are factors of $k+1$ or of $\frac{k+1}{2}$ according as k is even or odd [P. S. II (37)].

Let us consider the effect of these elements upon linear forms,

$$x_1u_1+x_2u_2+\ldots\ldots+x_nu_n,$$

with integer coefficients $x_1, \ldots\ldots, x_n$. This is transformed by $A_{1,\ldots\ldots,k+1}$ into the form

$$x_1'u_1'+x_2'u_2'+\ldots\ldots+x_n'u_n',$$

where

$$x_i'=x_i+\frac{k-1}{k+1}\sum_1^n x_h-(x_1+\ldots\ldots+x_{k+1}) \qquad (i=1, \ldots\ldots, k+1),$$
$$x_j'=x_j \qquad (j=k+2, \ldots\ldots, n).$$

Hence

$$\sum_1^n x_h'=\sum_1^n x_h+(k-1)\sum_1^n x_h-(k+1)(x_1+\ldots\ldots+x_{k+1}).$$

If therefore

$$(33) \qquad \sum_{h=1}^n x_h=(k+1)x_0 \ (k \text{ even}), \text{ or } \sum_{h=1}^n x_h=\frac{k+1}{2}x_0 \ (k \text{ odd}),$$

where x_0 is an integer, then also

$$\sum_1^n x_h'=(k+1)x_0' \ (k \text{ even}), \text{ or } \sum_1^n x_h'=\frac{k+1}{2}x_0' \ (k \text{ odd}),$$

where x_0' is also an integer. If we set

$$(34) \qquad y_0=x_0 \ (k \text{ even}), \text{ or } y_0=\frac{x_0}{2} \ (k \text{ odd}),$$

we find in either case that

$$(35) \quad \begin{cases} y_0'=y_0+\{(k-1)y_0-(x_1+\ldots\ldots+x_{k+1})\}, \\ x_i'=x_i+\{(k-1)y_0-(x_1+\ldots\ldots+x_{k+1})\} \qquad (i=1, \ldots\ldots, k+1), \\ x_j'=x_j \qquad (j=k+2, \ldots\ldots, n). \end{cases}$$

This linear transformation (35) is precisely the generator $A_{1,\ldots\ldots,k+1}$ of the group $g_{n,k}$. Hence

(36) *Under the group $e_{n,k}$ linear forms with integer coefficients x which satisfy the relations* (33) *are permuted contragrediently to integer linear forms under the group $g_{n,k}$.*

Hence within the totality of integer linear forms defined by (33) we shall have a modular theory identical with that of $g_{n,k}$ which has been discussed in the preceding sections for the modulus 2.

In the case of the ultra-elliptic set P^p_{2p+2} the single condition for self-association is

$$u_1+u_2+\ldots\ldots+u_{2p+2}=0.$$

This is a linear form included in the aggregate (33). It corresponds to the point 0, 1, 1, ..., 1 for the group $g^{(2)}_{n,k}$ or through the medium of the invariant polarized form M to the linear form $B_{1,2,\ldots,n}$. When p is odd, $B_{1,\ldots,n}$ is the invariant form L so that the requirement $B_{1,\ldots,n}=0$ leads to no change in the group $g^{(2)}_{2p+2,p}$. If, however, p is even, the form $B_{1,\ldots,n}$ (see cases $\varkappa=0$, $\nu=2$; $\varkappa=2$, $\nu=2$) is represented in the finite geometry by the point $P_{\alpha\beta}$, and we ask for the subgroup of $g^{(2)}_{2p+2,p}$ which leaves $P_{\alpha\beta}$ unaltered. When $p\equiv 0$, mod. 4, there is an invariant O quadric Q not on $P_{\alpha\beta}$ so that the group reduces to that of the O, E quadrics Q, $Q_{\alpha\beta}$. When $p\equiv 2$, mod. 4, there is an invariant O quadric $Q_{\alpha\beta}$ not on $P_{\alpha\beta}$ so that the group reduces to that of the O, E quadrics $Q_{\alpha\beta}$, Q. In either case the group is reduced from a GO_{2p+1} to a group with an invariant G_2 whose factor group is a GC_{2p-1}. Now if a group H has an invariant subgroup I with factor group F, and if F has an invariant subgroup I' with factor group F', then H has a larger invariant subgroup I'' which contains I whose factor group is F'. Hence we can state that

(37) *The group $g_{2p+2,p}$ (p odd) has an invariant subgroup whose factor group is GC_{2p-1} if $p\equiv 3$, mod. 4; or GO_{2p-1} if $p\equiv 1$, mod. 4. If p is even the group $g_{2p+2,p}$ has a subgroup g' which consists of those elements which leave $x_1+\ldots+x_{2p+2}$ unaltered, mod. 2, and g' has an invariant subgroup whose factor group is GC_{2p-1}.*

The above theorem concerning $g_{2p+2,p}$ can be translated at once to apply to the simply isomorphic Cremona group $G_{2p+2,p}$ which is projectively attached to the point set P^p_{2p+2}, the subgroup g' of the theorem being that which is determined by the projective conditions for self-association. Thus the fact that the self-associated set defines groups which are isomorphic with those of the half periods of the theta functions in p variables confirms the existence of a connection between the absolute invariants of the self-associated set and the theta modular functions. In case $p\equiv 1$, mod. 4, there is indicated further that in this connection an odd theta-characteristic is isolated.

BALTIMORE, *April* 15, 1918.